Closed
Self-intersection in a blow-up

M = Complex Surface

A (complex) blowup $Bl_p(M)$ replaces $p \in M$ with the set of complex lines in $T_pM \cong \mathbb{C}^2$, a set $\cong \mathbb{C}P^1$.

Blowup is characterized by: $\exists$ "blow-down map"

$$\begin{array}{c} Bl_p(M) \\ \downarrow \pi \\ M \end{array}$$

Such that

1. $\pi|_{Bl_p(M) - \pi^{-1}(p)}$ is an $\cong$ of complex mnflds.

2. $e := \pi^{-1}(p) \cong \mathbb{C}P^1 \subseteq Bl_p(M)$, called the exceptional divisor of the blowup.

Thus e is an embedded 2-sphere in the closed, oriented 4-manifold $Bl_p(M)$. as such, it has a self-intersection number $e \cdot e$, defined via:

Perturb e off itself to a surface $e' \subset M$ s.t. $e \pitchfork e'$.

Then $e^2 := e \cdot e :=$ the alg. intersection #
$$\sum_{z \in e \cap e'} \pm 1$$

<u>Proposition:</u> For any closed, complex surface M, and any $p \in M$,

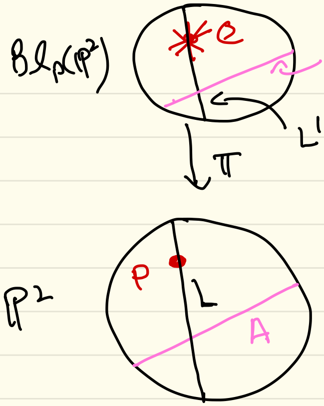
let $e =$ exceptional divisor of $Bl_p(M) \to M$.

Then the alg. int. # of e in M is
$$e^2 = -1 \; .$$

<u>Cor:</u> e is <u>rigid</u>: it cannot be homotoped to any complex $e' \subset M$ with $e \pitchfork e'$.

<u>Proof #1:</u> Do it by hand, using the fact
$$Bl_p(M) \underset{\text{diffeo}}{\cong} M \# \overline{\mathbb{CP}^2}$$

<u>proof #2</u> (for $M = \mathbb{P}^2$, for simplicity)



Choose lines ($\cong \mathbb{P}^1$) $L, A \subset \mathbb{P}^2$ s.t. $p \notin A$, $p \in L$

Let $L' \subset Bl_p(\mathbb{P}^2)$ be the <u>strict transform</u> of L:

$$L' := \overline{\pi^{-1}(L - p)}^{\,zar} \cong \mathbb{P}^1$$

- $[\pi^{-1}(A)] \cdot [e] = 0$ since $\pi^{-1}(A) \cap e = \emptyset$.
- $\pi^{-1}(L) = L' \cup e$ so
  $[\pi^{-1}(L)] = [L'] + [e]$
- $[L'] \cdot [e] = 1$

Now $[A] = [L] \in H_2(\mathbb{P}^2; \mathbb{Z})$ so

$$[\pi^{-1}(A)] = [\pi^{-1}(L)] = [e] + [L'] \qquad (*)$$

and so $\displaystyle 0 = [\pi^{-1}(A)] \cdot [e] \underset{\text{by } (*)}{=} ([e] + [L']) \cdot [e]$

$$= [e] \cdot [e] + [L'] \cdot [e]$$

$$= [e] \cdot [e] + 1$$

$\Rightarrow [e]^2 = -1$ $\square$